International Journal of Theoretical Physics, Vol. 31, No. 7, 1992

Axioms for Quantum Theory

Gerhard Gerlich[1]

Received August 13, 1991

The first three of these axioms describe quantum theory and classical mechanics as statistical theories from the very beginning. With these, it can be shown in which sense a more general than the conventional measure theoretic probability theory is used in quantum theory. One gets this generalization defining transition probabilities on pairs of events (not sets of pairs) as a fundamental, not derived, concept. A comparison with standard theories of stochastic processes gives a very general formulation of the non existence of quantum theories with hidden variables. The Cartesian product of probability spaces can be given a natural algebraic structure, the structure of an orthocomplemented, orthomodular, quasi-modular, not modular, not distributive lattice, which can be compared with the quantum logic (lattice of all closed subspaces of an infinite dimensional Hilbert space). It is shown how our given system of axioms suggests generalized quantum theories, especially Schrödinger equations, for phase space amplitudes.

1. INTRODUCTION

In rather long treatises written in German, I developed systems of axioms for classical mechanics and quantum mechanics (Gerlich, 1974, 1977). Later, I tested them in courses of lectures in theoretical physics (Gerlich, 1980, 1983, 1987*b*, 1991). In the meantime, I found modifications and simplifications in the formulation of the axioms. In particular, I could give a more general formulation for the nonexistence of quantum theories with hidden variables containing all nonexistence theorems known to me (Gerlich, 1977, 1987*b*).

If one tries to formulate the mathematical foundations of a physical theory axiomatically, one at once provokes the critical question of what *new* experimental results are predicted by this theory. On the other hand, if one presents an improved mathematical method to describe certain physical processes, one is asked to show how the conventional (worse) methods could

[1]Institut für Mathematische Physik der Technischen Universität Carolo-Wilhelmina, 3300 Braunschweig, Germany.

0020-7748/92/0700-1103$06.50/0

give the same results. Were this kind of criticism the exception and not the rule, one would not have to take any trouble with this. A system of axioms of quantum theory fulfils the task of laying down (restrict) the general properties of the mathematical structures of quantum theory. *It should just yield no other theory than quantum theory.* This does not mean, of course, that a system of axioms could not give hints where appreciable modifications of a theory are possible. It is for this very purpose that most such investigations are carried out. Moreover, the usefulness of an axiomatic formulation of a theory lies in the fact that the theory is summarized in a few sentences. This is well known to everyone who has given an axiomatic introduction to classical mechanics.

Axioms of a physical theory mean a restriction of the mathematical possibilities. They cannot be proven. But it is possible to make them clear or plausible. This means that one could try to show the usefulness and simplicity of the selected mathematical structures. For the arrangement of the axioms, this has as a consequence that the plausibility of the axioms should decrease with the progressive restriction of the mathematical possibilities. I hope that the given system of axioms satisfies this criterion.

Restricting the general mathematical properties, nobody will be successful with the attempt to include all properties completely. This means that, in principle, such a system of axioms can never be complete and one must always live with the questions: Did one disregard essential properties of quantum theory? Are important fields of the theory excluded by the mathematical restrictions? Is this theory, fixed by the general statements, really the whole quantum theory? Viewed from this standpoint, the new arrangements and formulation of the axioms should have advantages.

The axioms (A1)–(A3) contain classical mechanics and quantum theory. For Newtonian mechanics one can read them as "preaxioms" which were and are used more or less consciously because no other mathematical possibilities were taken into account. On the other hand, for quantum theory, it is necessary to write them down because they allow one to explain the differences from classical mechanics. With them, it is possible to formulate in what sense a more general than the usual measure-theoretic probability theory is used in quantum theory. This gives the relation to the quantum logical systems of axioms and a simple nonexistence statement for quantum theories with hidden variables: The usual theory of stochastic processes provides formulas for transition probabilities that are too special. Furthermore, these preaxioms (A1) and (A2) suggest a more likely justifiable formulation of the superposition principle of quantum theory (AQSP).

The subsequent axioms (AQS1) and (AQS2) are typical of quantum theory and are not valid for classical mechanics. It is possible to formulate the corresponding axioms for classical mechanics (including relativistic mechanics of noninteracting mass points). The axiom (AMN1) corresponds to the first and (AMN2) to the second Newtonian axiom. From this parallel, an estimation of the corresponding quantum mechanical axioms can be deduced. As little as the first two Newtonian axioms describe each specific mechanial system completely, so as little do the axioms (AQS1) and (AQS2) describe each quantum mechanical system. In particular, d'Alembert's and Gauss's principle are missing settling how constraints modify the equations of motion of classical mass points. As one can conclude the form of interactions from Newton's second law and from the Newtonian law of gravity, this is similarly possible with (AQS2) and Maxwell's equations in quantum theory. There exists a simple parallel formulation of these two axioms fitting Lorentz transformations and thus electrodynamics even better. These are the axioms (AQD1) and (AQD2) corresponding to Dirac's equation without and with external electric fields, respectively.

2. THE AXIOM (A1) AND THEORIES WITH HIDDEN VARIABLES

The first axiom summarizes two empirical facts:

(a) Every physical theory is finally tested by the reading of numbers of a scale (with error bounds). We call this the observation of events (in the decision), where events are elements of a class of subsets of a certain set; in this example the observed event is a certain interval of R^1 (real line).

(b) The value of such measured numbers is physically meaningless if one does not know how the experiment was performed. In particular, this knowledge could be given by an observed or an assumed event (in the condition): an event A which one knows to predict the probability of the event B. One could consider the motion of a car on an inclined plane or the movement of the planets. The measurements of the space and velocity coordinates alone are not physics. Physics begins with predicting the values at a later instant of time (with error bounds) with a model. Statements of measured values with error bounds are typical probability statements: For instance, with probability 0.999 the measured value should be found in a certain interval upon performing the experiment in the same way (the same event in the condition).

These plausibility arguments should be enough. For a more detailed discussion see Gerlich (1974, 1977, 1987*b*, 1991).

(A1) The statements of physics are statements about spaces of events. The statements about events are formulated with frequencies, normed contents, or probability measures for *pairs of events*: $q(A, B)$ is the probability of observing the event B if one knows the event A. $q(A, B)$ is called the transition probability.

Spaces of events are sets X, respectively Y, with σ-algebras **A**, respectively **B**. Elements of σ-algebras usually are called measurable sets or, in the measure-theoretic probability theory, events. In this sense, in (A1), very conventional concepts of the measure-theoretic probability theory founded by Kolmogoroff are used (Kolmogoroff, 1933; Bauer, 1974; Kingman and Taylor, 1966). Only the concept of the transition probability $q(A, B)$ is introduced as an additional fundamental concept and not as a derived concept. In conventional probability theory, with two spaces of events $(X, \mathbf{A})$ and $(Y, \mathbf{B})$, one constructs a new common space of events $(X \times Y, \mathbf{A} \otimes \mathbf{B})$. $X \times Y$ is the Cartesian product of the sets X and Y, $\mathbf{A} \otimes \mathbf{B}$ is the product-σ-algebra generated by the sets $A \times B$ with $A \in \mathbf{A}$, $B \in \mathbf{B}$ (Figure 1). With a probability measure on this space one calculates the transition probability as a *conditional* probability. The events A are replaced by $A \times Y$, the events B are replaced by $X \times B$. The smallest σ-algebra containing $\{A \times B \mid A \in \mathbf{A}, B \in \mathbf{B}\}$ is $\mathbf{A} \otimes \mathbf{B}$. With a probability measure P on $\mathbf{A} \otimes \mathbf{B}$ one calculates the transition probability as a conditional probability

$$q_{\mathrm{CL}}(A, B) = \frac{P(A \times B)}{P(A \times Y)}$$

For disjoint sets A and A' the Cartesian products $A \times B$ and $A' \times B$ are disjoint (Figure 1). Therefore one gets

$$P((A \times B) \cup (A' \times B)) = P(A \times B) + P(A' \times B), \qquad A \cap A' = \varnothing$$

$$q_{\mathrm{CL}}(A \cup A', B) = \frac{P((A \cup A') \times B)}{P((A \cup A') \times Y)}$$

$$= \frac{P((A \times B) \cup (A' \times B))}{P((A \times Y) \cup (A' \times Y))}$$

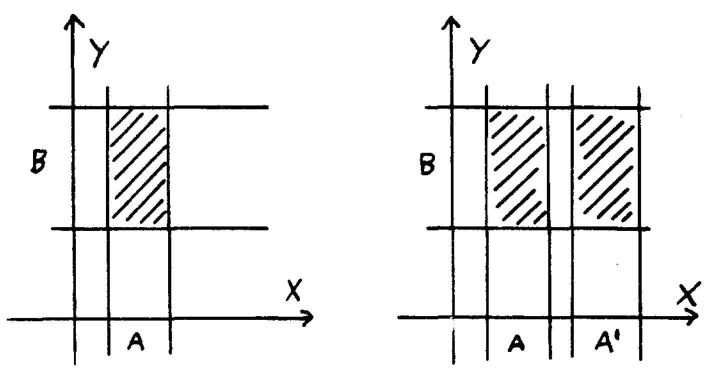


Fig. 1

$$=\frac{P(A\times B)}{P(A\times Y)+P(A'\times Y)}+\frac{P(A'\times B)}{P(A\times Y)+P(A'\times Y)}$$

$$=\lambda_1 q_{\mathrm{CL}}(A, B)+\lambda_2 q_{\mathrm{CL}}(A', B)$$

with $\lambda_1+\lambda_2=1$ and $0\leq\lambda_1,\ \lambda_2\leq 1$.

This simply means that, for $q(A, B)=a$, $q(A', B)=b>a$, $A\cap A'=\varnothing$, $q(A\cup A', B)$ can only have values in the interval $[a, b]$; if a, b are unequal to zero, the combined transition probability cannot be zero (no interference). One gets the same result if (A, B) can be read as $A\wedge B$ of a common probability space defining $q_{\mathrm{CL}}(A, B)=P(A\wedge B)/P(A)$. Therefore one gets for classical transition probabilities:

(CL) The transition probability of a union of disjoint events in the condition is a convex linear combination of the individual classical transition probabilities.

This statement is nothing but a modification of Bayes' formula of conventional probability calculus (Bauer, 1974, p. 134; Kingman and Taylor, 1966, p. 274). The statement (CL) (short of for convex linear combination or classical) remains true for an apparently considerably more general formula for classical transition probabilities (Gerlich, 1977, 1987*b*):

Let $x_t^i(\omega)$ be a family of stochastic processes on *one* probability space (Ω, A, P). One could think of a part of the solution curves of a system of ordinary differential equations describing a system of mass points. $\omega\in\Omega$ are the initial values of all points with a given probability distribution P. In this situation one can take as a rather general formula for a transition probability

$$\bar{q}_{\mathrm{CL}}(A, B)=\frac{\Sigma_{i,j}\, w_{ij} P(x_{t_1}^{i^{-1}}(A)\cap x_{t_2}^{j^{-1}}(B))}{\Sigma_i\, w_i P(x_{t_1}^{i^{-1}}(A))}$$

with $0\leq w_{ij}\leq 1$, $w_i=\Sigma_j\, w_{ij}$, and $\Sigma_i\, w_i=1$.

With this formula, too, one can prove the statement (CL) (Gerlich, 1987*b*, p. 142). Anticipating the result given later in this paper that the formula for quantum mechanical transition probabilities is more general, in the sense that (CL) need not be true, we get our statement of the nonexistence of quantum theories with hidden variables: A family of stochastic processes on one probability space produces too special transition probabilities. The whole point of this statement is to be seen in the fact that the richness of the structure and the exactness of the quantum mechanical formula, which has interference terms, is testably superior to the classical formula. Not certain unaccuracy effects, but the fantastically exact probability predictions are typical of quantum theory. This fits the historical experience that very exact measurements (spectral lines) were the reason for modifying classical mechanics and electrodynamics.

Essential for these proofs of (CL) is the fact that the family of stochastic processes is defined on *one* probability space. Deriving Bell's inequalities, this one probability space is used for the calculation of the classical expectation values of the product of two noncommuting observables (see Bell, 1965; and Jauch, 1973, but also Barut and Meystre, 1984). With the model of stochastic processes one cannot reproduce our formula for the transition probability of quantum theory, no matter how extensive the one probability space (Ω, P) of the hidden variables is chosen. For these considerations it is not necessary to assume the differentiability of the paths $x_t^i(\omega)$ with respect to t. Most theories with hidden variables for quantum theory known to me fall under the given nonexistence statement, especially the one circulated by the mathematician E. Nelson (1967, p. 116). How can it be that such an excellent mathematician like Nelson, who knows the discussed difficulties (Nelson, 1967, p. 129), can support such an insufficient theory? First of all, one could deny the general validity of our formula given later in (A2). But the interference effects of quantum theory are a generally accepted property of quantum-theoretic probability theories. Therefore one needs a better explanation. At a first glance the differential equation for the time dependence of the probability density f_t

$$\frac{\partial f_t}{\partial t} + L(f_t) = 0$$

looks like the corresponding equation of a (Brownian) stochastic process. Then one thinks that one has found the theory of hidden variables. But nobody can prove that this equation uniquely determines one stochastic process. For instance, one can define with densities $f_t(x)$ the product measure over all t. Or one solves the equation with a Green's function $f_t(x_0, x)$ and uses the latter as the density of the transition function of a second Markovian process. The latter is surely different, as the first has a trivial transition function. Usually only for Markovian processes can one conclude the corresponding stochastic process from the differential equation for the probability density of the transition function if the initial value problem for the Green's function can be completely solved. But in general, one cannot conclude from such a differential equation for a probability density that the process producing this equation was a Markovian process. Therefore, instead of using one formula for a transition probability, one has to adopt many formulas for each special physical situation.

Even if one uses an independent second probability space for the starting points, one gets (CL) with a general transition function of a general stochastic process:

$$q_{\mathrm{CL}}(A, B) = \frac{\int_A P_{t_0}(dx_0)\, P(t_0, x_0\,; t, B)}{\int_A P_{t_0}(dx_0)}$$

With $A \cap A' = \varnothing$, one gets

$$N = \int_{A \cup A'} P_{t_0}(dx_0) = \int_A P_{t_0}(dx_0) + \int_{A'} P_{t_0}(dx_0)$$

$$\begin{aligned} &q_{\mathrm{CL}}(A \cup A', B) \\ &= \frac{\int_{A \cup A'} P_{t_0}(dx_0)\, P(t_0, x_0\,; t, B)}{N} \\ &= \frac{\int_A P_{t_0}(dx_0)\, P(t_0, x_0\,; t, B) + \int_{A'} P_{t_0}(dx_0)\, P(t_0, x_0\,; t, B)}{N} \\ &= \lambda_1 q_{\mathrm{CL}}(A, B) + \lambda_2 q_{\mathrm{CL}}(A', B) \end{aligned}$$

$$\lambda_1 = \int_A P_{t_0}(dx_0)/N, \qquad \lambda_2 = \int_{A'} P_{t_0}(dx_0)/N$$

$$0 \le \lambda_1, \lambda_2 \le 1, \qquad \lambda_1 + \lambda_2 = 1$$

The statement (CL) can be proven even if the stochastic process is defined by the n-time distribution functions satisfying Kolmogoroff's compatibility conditions. With the theorem of Kolmogoroff (Bauer, 1974, p. 349; Kingman and Taylor, 1966, p. 381), a canonical stochastic process on one probability space exists reproducing the same n-time probability distributions. This space is the set of all paths (time functions) of the process. Thus, one has constructed a space of hidden variables for the n-time distribution functions which can be used to prove (CL). Therefore, all theories with hidden variables that use the theory of stochastic processes cannot use the standard formula of stochastic processes for quantum mechanical transition probabilities, as it does not give interference terms.

This exposition also shows that one should not be surprised if the models of quantum theory and general relativity theory do not fit together if the latter is founded on the notion of classical curves of particles.

Approximately one can always use a classical statistical description, for instance, substituting the noncommuting operators of quantum theory by approximate commuting operators (Davies, 1976, Chapter 3; Holevo, 1973; von Neumann, 1932/1968, p. 215). With the corresponding probability distributions one gets no interference effects for the conditional probabilities. Our probability formulas of the quantum mechanical models have more structure. Often one states the insufficiency of the classical models in quantum theory in that they admit dispersion-free ensembles. This immediately suggests that the quantum mechanical models were particularly inexact. But just the opposite is true. Of course, one can in principle make very precise probability predictions with classical models. But, in practice, one does not use these precise predictions because they are surely bad! Dispersion-free ensembles in the mathematical model cannot guarantee an exact description of nature. Just because the models of classical mechanics allow such a precise description, they are worse than the models of quantum mechanics.

In connection with these considerations, one should warn about an overinterpretation. The given nonexistence statement tells us which kind of theory, the standard theory of stochastic processes or probability theory, is not sufficient to get the quantum mechanical formulas for the transition probabilities. By no means does this statement say that in some sense the quantum mechanical description of nature is perfect or could not be improved. Many people overlook this point when discussing the nonexistence statement of von Neumann. Von Neumann could not and did not try to exclude an improvement, that is, a change of the known quantum theory, with his nonexistence statement (von Neumann, 1932/1968, p. 109, 171). He only excluded a certain ensemble theory, *accepting the given mathematical structure of the existing quantum theory*. In my diction this ensemble theory is a rather specific theory of stochastic processes. Theories with hidden variables giving formulas contradicting quantum theory or enlarging the applicability of quantum theory can be given many names, but with certainty they are not theories with hidden variables *for quantum theory* (compare, however, Belinfante, 1973). This remark also applies to the "theories with hidden variables" that add averaging procedures to the standard apparatus of quantum theory (Gudder, 1970; Pancović, 1989) or our generalized Schrödinger equation on phase space probability amplitudes (Dietert, 1990). In my opinion, it is a rather cheap trick to show that all nonexistence statements for theories with hidden variables for quantum theory are wrong if one changes quantum theory especially adding a structure nobody uses in practice (Gudder, 1970).

A typical performance of theories with hidden variables is to make calculations that, with the existing quantum theory, cannot be made or are less exact. Therefore, our nonexistence statement cannot mean that in future a more "microscopic" theory will not be possible for certain applications than the existing quantum theory is. With a similar standpoint, one would never have found quantum theory after classical mechanics. With respect to our considerations, the main difference between quantum theory and classical mechanics is that the mathematical models of quantum theory give more exact probability distributions. Therefore it is clear where future progress of the theory should be looked for: This should be a theory which gives even more exact probability distributions. Only the never proved and never provable assertion that the formulas of classical mechanics have these properties produces a fear of such theories. If our axioms (A1)–(A3) have some universal validity, they should survive such a progress of the theory because they are as valid in classical mechanics as in quantum mechanics. The way to mathematical models giving more exact probability distributions is not spoiled by our nonexistence statement. But one should not use the theory of stochastic processes for calculating transition probabilities, as it is even now too poor because of (CL).

3. THE CARTESIAN PRODUCT OF PROBABILITY SPACES AND THE QUANTUM LOGIC

The axiom (A1) contains a certain algebraic structure which can be worked out. I called this structure parametrized probability spaces, or parametrized spaces of events (Gerlich, 1977, 1981, 1987*b*). The probability statements mentioned in (A1) are statements about pairs of events (A, B), thus elements of the Cartesian product $\mathbf{A} \times \mathbf{B}$ of the σ-algebras $\mathbf{A}$ and $\mathbf{B}$. Unlike classical probability theory, the pairs of events (A, B) are not automatically identified with the Cartesian products $A \times B$ of the events (sets). Usually incompatible events are disjoint sets. Therefore, one could ask how one could define the incompatibility of pairs of events. Such a situation is given in physical practice if an experimental physicist says that she cannot verify the experiment of a colleague because the results of her experiment, which is of course much more precise, exclude the results of her colleague. For instance, the second had measured 2π with error bounds excluding π, whereas the first had measured π with error bounds excluding 2π. By tacit agreement, one assumes, of course, that the second experimental physicist had made the same experiment described by the first. With our formulation of (A1) this means: If it should make sense that the events in the decision B and B' exclude themselves, then necessarily the event A in the condition should be the same. We call this the principle of meaningful comparison:

(MC) Pairs of events are comparable iff the events in the condition are equal.

With this, one can define in $\mathbf{A} \times \mathbf{B}$ a partial ordering $\trianglelefteq$:

(PO) $(A, B) \trianglelefteq (A', B')$ iff $A = A'$ and $B \cap B' = B$.

It is easy to check that $(\mathbf{A} \times \mathbf{B}, \trianglelefteq)$ is a partially ordered set. In this set, one can define an orthocomplementation $\perp$:

(POC) $(A, B)^{\perp} = (A, \complement B)$.

In the usual way, two elements of $\mathbf{A}\times\mathbf{B}$ are called orthogonal or disjoint iff

$$(A, B) \trianglelefteq (A', B')^{\perp} = (A', \complement B'), \qquad \text{resp. } A = A' \text{ and } B \subset \complement B'$$

In this orthocomplemented partially ordered set $(\mathbf{A}\times\mathbf{B}, \trianglelefteq)$, in general, two pairs of events do not have an infimum or supremum, namely, if the events in the condition are not identical. With the following trick this can be avoided. One identifies all pairs if the event in the decision is the empty set (impossible event) and call it the element 0 of $\mathbf{A}\times\mathbf{B}$. Then one has to identify all pairs if the event in the decision is the total space Y (sure event) and one calls it 1. Then 0 is the smallest and 1 is the greatest element of this partially ordered set with $0^{\perp}=1$ and $1^{\perp}=0$. If two pairs of this partially ordered set are not comparable because the events in the condition are not equal, then their infimum is 0 and their supremum is 1. Because now two pairs of events of this partially ordered set $(\mathbf{A}\times\mathbf{B}, \trianglelefteq)$ have an infimum and a supremum, one can turn it into a lattice in the usual way, defining $\wedge$ as the infimum and $\vee$ as the supremum. This lattice $(\mathbf{A}\times\mathbf{B}, \trianglelefteq, \wedge, \vee)$ is orthocomplemented, orthomodular, quasimodular, not modular, not distributive [for a more complete discussion see Gerlich (1987*b*)]. Depending on the σ-algebra $\mathbf{B}$, this lattice can be atomic and complete. For instance, the Boolean σ-algebra of the Lebesgue-measurable sets of R^1 is atomic; if one takes the equivalence classes of sets having difference sets with Lebesgue measure zero, the σ-algebra is not atomic $[A \sim A' : \mu_L(A \,\triangle\, A') = 0]$. Therefore, this lattice $\mathbf{A}\times\mathbf{B}$ has all the important general properties usually stated for a quantum logic, the lattice of all closed linear subspaces of an infinite-dimensional Hilbert space. Of course, this lattice $(\mathbf{A}\times\mathbf{B}, \trianglelefteq, \wedge, \vee)$ cannot have all the properties given by Piron (1976). If I am right, only the covering law is missing (Piron, 1976, p. 24). This law is also missing in the just mentioned Lebesgue-σ-algebra of the equivalence classes. This lattice $(\mathbf{A}\times\mathbf{B}, \trianglelefteq, \wedge, \vee)$ is not identical with the corresponding sublattice of the closed linear subspaces of a Hilbert space (quantum logic). The problems can best be illustrated with the projection-valued measures P_A and $\tilde{P}_B$ belonging to canonical conjugate momentum and position operators. In quantum logic the "and" of P_A and $\tilde{P}_B$ gives the null space. Therefore, one cannot use a probability measure on the quantum logic to define a nontrivial transition probability using $P(A \wedge B)/P(A)$. In standard approaches one can add to the projections, which are the elements of the quantum logic, the positive operators $P_A \tilde{P}_B P_A$ to calculate transition probabilities. Our events in the condition determine the probability distribution (the states). With respect to the events in the decision, $q(A, B)$ is a standard measure-theoretic probability distribution; the event A only is an index for this probability distribution defined on the Boolean σ-algebra $\mathbf{B}$. There is added a new "and," namely (A, B) between events in the condition and decision. If one combines events in the condition, interference can occur.

These considerations should not mean that, with our axiom (A1) and our principle of meaningful comparison (MC), the structure of the lattice of all closed linear subspaces of an infinite-dimensional Hilbert space is excluded from quantum theory. The following axiom (A2) restricts the natural vector spaces of the events to concrete Hilbert spaces. These Hilbert spaces have their lattice of closed linear subspaces whether we like designating them explicitly or not. In our system of axioms, quantum logic is a consequence of (A2) without giving special significance to the lattice properties of the closed subspaces. In a quantum logical approach, the Hilbert space structure results as a consequence of the plausible introduced lattice properties. But our example shows that the essential difficulty is less to make plausible the above-mentioned general properties of a lattice that our primitive lattice $(\mathbf{A} \times \mathbf{B}, \trianglelefteq, \wedge, \vee)$ has already to find plausible arguments for the supremum and infimum of nonorthogonal linear subspaces than it is. Perhaps our example could illustrate the significance of the covering law of quantum logical approaches.

Apparently the essential point of our approach is to be seen in the distinction of the events in the condition and decision. In the representation of Ludwig (1976) this distinction is given by separating an experiment into preparation and registration (effect) parts. In the representations following von Neumann (1932/1968) and (Mackey, 1963; Jauch, 1973; Piron, 1976), one can find this distinction between the concepts "states" and "properties, propositions, questions." Though these approaches fit our system of axioms without great difficulties [for a more explicit discussion see Gerlich (1987*b*), Chapter III], an essential point should not be overlooked. With (A1) it is clear where the experimental test of a theoretical model should take place: One has to check certain predicted (calculated) probability distributions for sets. Taking as an additional fundamental concept the transition probability, which is not derived from a probability, the mathematical structure is sufficiently general to include the interference effects of quantum mechanical probability theories. With this approach the expectation value is a derived concept corresponding to the situation in classical probability theory. This leads to the introduction of general self-adjoint operators in addition to the projectors (Gerlich, 1987*b*, p. 113). But there is another possibility to generalize classical probability theory taking as a fundamental concept the expectation value. In conventional quantum theory one can interpret the expectation values as certain positive linear functionals on certain sets of operators (algebras of operators). Working out such an axiomatic system [instead of (A1)], one has to give plausible arguments for these sets of operators and the admissible positive linear functionals. As long as these sets of operators contain the projection-valued measures (PV-measures) mapping measurable sets (events) of a σ-algebra into the projections of a Hilbert space in a certain way (Davies, 1976, p. 35; Ludwig, 1976, p. 432; Gerlich, 1987*b*, p. 90), one, at least formally, gets our given "experimental windows" of the theory corresponding to (A1) if one describes the dependence of the events in the condition with the language of the "states": The expectation of the projection operator P_B in the state ρ is just the probability of the event B. In such an approach one has to introduce the transition probabilities as a derived concept. This is possible generalizing the PV-measures to positive operator valued measures (POV-measures), at least if the space of events in the condition is discrete (Davies, 1976, p. 15; Ludwig, 1976, p. 432; Gerlich, 1987*b*, p. 125). When this structure of a PV-measure and POV-measure is incorporated in the generalization of the expectation, it is in agreement with our system of axioms. But without such a structure the statement that probabilities are assigned to events (sets) is meaningless.

4. THE NATURAL HILBERT SPACES AND THE AXIOM (A2)

Formulating the axiom (A2), one needs a few technical definitions. Assume measures ν and μ on the σ-algebras **A** and **B**, respectively, for the spaces of events $(X, \mathbf{A})$ in the condition and $(Y, \mathbf{B})$ in the decision. These measures should characterize the events that could have a positive probability. This means the following. If one would like to show with the mathematical model that it is impossible to measure a prescribed number but only a value in a finite interval with a positive probability, one could use the Lebesgue measure on the Lebesgue-σ-algebra of the real line. If one would like to fix conventionally that the measured value should be a certain number (for instance, n, π), one could take the counting measure for these numbers. One could think of the spectra of self-adjoint operators, the discrete spectra are characterized by the counting measure, the continuous spectra by the Lebesgue measure. The measures ν and μ are assumed to be σ-finite. Apparently the finite complex linear combinations of the characteristic functions of the events with finite measure are a vector space of functions E over the field of complex numbers. We denote the quotient space E/N, the subspace N spanned by the characteristic functions of sets with measure zero, by the vector space of events in the condition or decision (Table I).

In another way one can formally check whether such an elementary function is an element of the natural vector space of events. Let f be a nonnegative function defined on the positive real numbers being zero only at the origin:

$$f(x)>0 \quad \text{for } x>0, \qquad f(0)=0$$

Table I

Condition	Decision
$X, \mathbf{A}, \nu$	$Y, \mathbf{B}, \mu$
$\varphi = \sum_{j(<\infty)} \alpha^i \chi_{A^i} \in E$	$\psi = \sum_{j(<\infty)} \beta^j \chi_{B^j} \in E$
$\nu(A^i) < \infty,\ A^i \cap A^j = \varnothing\ (i \neq j)$	$\mu(B^j) < \infty,\ B^i \cap B^j = \varnothing\ (i \neq j)$

For example, such a function is $f(x) = x^p$ or $x/(1+x)$. The integral

$$\int f(|\psi(y)|)\mu(dy) = \int \sum_j f(|\beta^j|) I_{B^j}(y)\mu(dy)$$
$$= \sum_j f(|\beta^j|)\mu(B^j)$$

is finite iff ψ is an element of the natural vector space and zero iff ψ represents the null vector. One should pay attention to the integral not being the usual $\mathscr{L}^p$-norm in the case $f(x) = x^p$ $(p \geq 1)$, but the pth power of the norm. In the case $f(x) = x^p$, $0 < p < 1$, and $f(x) = 1/(1+x)$ it is well known that $\int f(|\psi - \varphi|)\mu(dx)$ is a metric for the natural vector spaces. It is a standard technique in mathematics to complete these spaces with respect to the metrics and to work with the completed spaces. This simplifies calculations analogously to substituting difference equations by differential equations. By this technique no experimentally testable mistakes occur if one uses sensibly chosen measures. In the case $f(x) = x^2$ $(f(x) = x^p,\ 1 \leq p \neq 2)$ the corresponding space $\mathscr{L}^2$ $(\mathscr{L}^p)$ is a Hilbert space (Banach space) with scalar product

$$\langle \psi | \varphi \rangle = \int \bar{\psi}(y)\varphi(y)\mu(dy)$$

(norm $\|\psi\| = [\int |\psi|^p \mu(dy)]^{1/p}$). We call these spaces *natural Hilbert spaces* (Banach spaces) of the events in the condition or decision.

Only the characteristic functions of events with finite measures are elements of these natural vector spaces. For the characteristic function of an event B with not necessarily finite measure, one can define by

$$P_B \psi|_y = \chi_B(y)\psi(y) = \sum_j \beta^j \chi^{(y)}_{B \cap B^j}$$

an idempotent bounded linear operator P_B from the natural vector space into itself. In the case of the natural Hilbert space, this operator is self-adjoint and a projector. The events with finite measure can therefore be represented by an element of the vector space or by a projector. With the function f, define

$$\mu^f_{\psi,\mu}(B) = \int f(|P_B\psi|_y|)\mu(dy)$$

By a simple calculation one sees that $\mu^f_{\psi,\mu}$ defines a *finite* measure on **B** depending on ψ, f, μ. In the natural Hilbert space, one can write

$$\mu_{\psi,\mu}(B) = \langle\psi|P_B\psi\rangle = \langle P_B\psi|P_B\psi\rangle$$

Such a finite measure can be normed,

$$P^f_{\psi,\mu}(B) = \frac{\int f(|P_B\psi|_y|)\mu(dy)}{\int f(|\psi(y)|)\mu(dy)}$$
$$= \frac{\int \chi_B(y) f(|\psi(y)|)\mu(dy)}{\int f(|\psi(y)|)\mu(dy)}$$

respectively in the natural Hilbert space

$$P_{\psi,\mu}(B) = \frac{\langle\psi|P_B\psi\rangle}{\langle\psi|\psi\rangle}$$

Thus we have found a candidate for a formula for the transition probability $q(A, B)$ mentioned in (A1). All that is missing is the dependence on the event A in the condition. f and μ should be independent of A, thus only ψ remains:

All these considerations on the natural vector spaces of the events would be superfluous if, for certain classes of ideal measurements, one could not describe the dependence on the event A in the condition by linear maps of the natural vector spaces:

(D) We call a model for a physical experiment a L^f-linear (L^p-linear, unitary) pair if it is possible to write the formula of the transition probability in the form

$$q(A, B) = \frac{\mu^f_{\psi_A,\mu}(B)}{\mu^f_{\psi_A,\mu}(y)} = \frac{\int \chi_B(y) f(|\psi_A(y)|)\mu(dy)}{\int f(|\psi_A(y)|)\mu(dy)}$$

with

$$\psi_A = L(\chi_A)$$

resp.

$$q(A, B) = \frac{\langle U(\chi_A)|P_B U(\chi_A)\rangle}{\langle U(\chi_A)|U(\chi_A)\rangle}$$

L and U being linear, respectively unitary, maps of the natural vector spaces (Hilbert spaces).

The following axiom restricts these possibilities [Mielnik (1974), however, proposed $\mathscr{L}^p$-spaces with $p \neq 2$ for generalized quantum theories with nonlinear Schrödinger equations]:

(A2) General probability distributions are given by (convex linear combinations of) the transition probabilities of a unitary pair.

In (A2) the convex linear combinations mean the transition from the pure states to the mixed states or density operators. Changing the measure μ is equivalent to allowing in (D) positive functions on A instead of the characteristic functions (Gerlich, 1987*b*, Chapter II.12).

The axiom (*A*2) *lays down where one has to look for the physical laws: The principles for the construction of the unitary maps U* (*with the natural Hilbert spaces*).

The natural Hilbert spaces have more structure than the vector space structure alone; they are integration spaces ($\mathscr{L}^2$-spaces). Therefore, our plausibility arguments leading to (A2) should suggest a strengthening taking into account this additional structure. In its strict sense this strengthening is no longer valid for classical mechanics. It is a formulation of Dirac's superposition principle suggested by our system of axioms:

(AQSP) The unitary maps of the unitary pairs can be written as integral transformations of the natural Hilbert spaces of the events.

The kernel of the integral transformation should be absolutely continuous with respect to the measures ν and μ not being the case for the models of classical mechanics. One should note that our formulation of Dirac's superposition principle does not contain the assumption that *all* operators of the mathematical model could be written as integral transformations, which was criticized by von Neumann in Dirac's representation of the mathematical structure of quantum theory (von Neumann, 1932/1968, esp. p. 14). Restricting this property to the operators of the unitary pairs, I consider this assumption more likely justifiable (Gerlich, 1987*b*, p. 175). Some remarks illustrate the situation given by (A2).

(a) The ψ_A are the conventional probability amplitudes and a typical unitary pair is given by the energy (with other quantum numbers) and the space

$$U(\chi_{\{E_i\}}(|_{\mathbf{r}} = \psi_{E_i}(\mathbf{r})$$

$|\psi_{E_i}(\mathbf{r})|^2$ is the probability density measuring the space coordinates of the particle if one knows the system having the eigenvalue of energy E_i. One can find this formulation already in Hilbert *et al.* (1927). Usually this formulation is restricted to the discrete spectrum (Davies, 1976, p. 15; Gerlich, 1987*b*, Chapter III.2a). This distinction between the discrete and continuous spectra should not occur in an appropriate mathematical model describing nature (Gerlich, 1977, Chapter 9). Therefore, I showed how the space could be made discrete without producing testably distinct physical statements in the sense of (A1) (Gerlich, 1977, p. 105). Mathematically this means the Lebesgue measure is to be the infinite approximation of the counting measure of equidistant points, the distance being small enough, which is well known from the common transition from the Fourier series to the Fourier integral. This is just the mathematical connecting link between the spectral theorems of M. H. Stone and H. Wintner (Hopf, 1937/1970, p. 18). Our formula for $q(A, B)$, in principle, is valid for the continuous spectrum, too. $\psi_{\Delta E}$ are the so-called eigenpackets (Hellwig, 1964, p. 144; Gerlich, 1987*b*, Chapter III.2a).

(b) In conventional representations of quantum theory, the map U describes the change of the "representation." Usually one assumes that, given all possible maps U, one could identify all natural Hilbert spaces. These identified spaces should be the "abstract" Hilbert space of the physical system. This identification produces some problems, the maps U being multivalued because of gauge transformations. The diagram with directly calculated U_1, U_2, U_3 need not be commuting (Gerlich, 1987*b*, p. 117) (Figure 2). In the case $\mathbf{A} = \mathbf{B}$ and U being time dependent, U is the Feynman propagator; for scattering experiments, the S-matrix is a map U in the sense of (A2).

(c) In classical mechanics the time-dependent U is given by a family of one-to-one mappings h_t of elements of the set X, the solutions of a system of ordinary differential equations: $x_t = h_t(x_0)$. x_t are the space and velocity

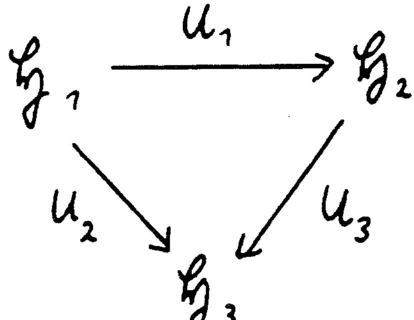


Fig. 2

coordinates of all particles at time t. Even if Liouville's theorem is not valid, the corresponding U_t is unitary with respect to a time-independent reference measure, setting

$$U_t(\psi)|_x = [\rho(x, t)]^{1/2}\psi(h_t^{-1}(x))$$

$\rho(x, t)$ is the density of the image measure generated by the flow h_t with respect to the measure at time zero, symbolically written as (Gerlich, 1987*b*, p. 101)

$$\mu(h_t^{-1}(dx)) = \rho(x, t)\mu_0(dx) = \mu_{h_t}(dx)$$

Progress in the ergodic theory of classical mechanics has been made possible by defining one-parameter families of unitary operators with the solutions h_t of the classical equations of motion and studying formulas of transition probabilities of a kind given in (A2). For these investigations, the measure structure of the phase space was essentially, showing that implicitly the axioms (A1) and (A2) have already been used for a long time in classical mechanics (Hopf, 1937/1970). For these classical models, the formula for the transition probability is independent of the selection of the function f. Therefore, this dependence on the selection of f as $f(x) = x^2$ is typical of quantum theory (Gerlich, 1974, 1977, 1987*b*).

(d) The formula for $q(A, B)$ given in (A2) can have interference terms absent in classical models. Assume A and A' are disjoint; then $\chi_{A\cup A'} = \chi_A + \chi_{A'}$ and $\langle U(\chi_A)|U(\chi_{A'})\rangle = 0$, U being unitary. One gets

$$q(A \cup A', B) = \frac{\langle U(\chi_{A\cup A'})|P_B U(\chi_{A\cup A'})\rangle}{\langle U(\chi_{A\cup A'})|U(\chi_{A\cup A'})\rangle}$$

$$= \frac{\langle U(\chi_A)|P_B U(\chi_A)\rangle}{N} + \frac{\langle U(\chi_{A'})|P_B U(\chi_{A'})\rangle}{N}$$

$$+ 2\,\mathrm{Re}\left(\frac{\langle U(\chi_{A'})|P_B U(\chi_A)\rangle}{N}\right)$$

$$q(A \cup A', B) = \lambda_1 q(A, B) + \lambda_2 q(A', B)$$

$$+ 2\,\mathrm{Re}\left(\frac{\langle U(\chi_{A'})|P_B U(\chi_A)\rangle}{N}\right)$$

$$N = \langle U(\chi_A)|U(\chi_A)\rangle + \langle U(\chi_{A'})|U(\chi_{A'})\rangle$$

$$\lambda_1 = \frac{\langle U(\chi_A)|U(\chi_A)\rangle}{N}, \qquad \lambda_2 = \frac{\langle U(\chi_{A'})|U(\chi_{A'})\rangle}{N}$$

$$0 \le \lambda_1, \lambda_2 \le 1, \qquad \lambda_1 + \lambda_2 = 1$$

The interference term distinguishes this term from a convex linear combination. Because of $\chi_{h_t(A)}(x)=x_A(h_t^{-1}(x))$, h_t invertible, one has $h_t(A)$ and $h_t(A')$ are disjoint for disjoint sets A and A'. Thus,

$$\chi_B(x)\rho(x,t)\chi_A(h_t^{-1}(x))\chi_{A'}(h_t^{-1}(x))$$
$$=\chi_B(x)\rho(x,t)\chi^{(x)}_{h_t(A)\cap h_t(A')}=0$$

Therefore, the interference term for systems of classical mechanics is zero. The following example shows how general the quantum mechanical formula could be. One could think of the slit experiment or the Stern–Gerlach experiment (Gerlich, 1977, p. 63; Gerlich, 1983, p. 240):

$$A\cap A'=\varnothing,\qquad A\cup A'=X,\qquad B\cap B'=\varnothing,\qquad B\cup B'=Y$$
$$\mu(B)=\mu(B')=v(A)=v(A')=\tfrac{1}{2}$$

Defining

$$U_\varphi(\chi_A)=\cos\varphi\,\chi_B+\sin\varphi\,\chi_{B'}$$
$$U_\varphi(\chi_{A'})=-\sin\varphi\,\chi_B+\cos\varphi\,\chi_{B'}$$

one gets

$$q(A,B)=\cos^2\varphi,\qquad q(A',B)=\sin^2\varphi,\qquad q(A,B')=\sin^2\varphi$$

In

$$q(A\cup A',B)=\tfrac{1}{2}-\tfrac{1}{2}\sin 2\varphi$$

the second term is the interference term. Setting $\varphi=\pi/4$, one gets $q(A\cup A',B)=0$.

An arbitrary convex linear combination of $q(A,B)$ and $q(A',B)$ results in

$$\lambda_1 q(A,B)+\lambda_2 q(A',B)=\lambda_1\cos^2\frac{\pi}{4}+\lambda_2\sin^2\frac{\pi}{4}=\frac{1}{2}$$

which surely is positive.

If one tries to choose suitable λ_1 and λ_2 to get $q(A\cup A',B)$, namely

$$\lambda_1\cos^2\varphi+(1-\lambda_1)\sin^2\varphi=\tfrac{1}{2}(1-\sin 2\varphi)$$

λ_1 should be given by $\lambda_1=\tfrac{1}{2}(1-\tan 2\varphi)$. Thus, only for $0\leq|\varphi|\leq\pi/8$ does one get for λ_1 a value between 0 and 1. Therefore, for U_φ with $\varphi>\pi/8$ one cannot find a λ_1 between 0 and 1 such that $q(A\cup A',B)$ can be written as a convex linear combination of $q(A,B)$ and $q(A',B)$.

Hence (CL) is not generally valid for the quantum mechanical transition probabilities. Note that already for this simplest example one must perform at least three experiments testing the interference term. Therefore, in reality, the two-slit experiment is three experiments (Gerlich, 1981; Gerlich, 1987*b*, p. 145).

The important point is that one can use *one* formula for the transition probability. If one uses different formulas for each experiment, no difficulties with (CL) can occur and therefore hidden variables theories are always possible [like those defined by Gudder (1970)].

5. THE TIME DEPENDENCE OF UNITARY PAIRS AND AXIOM (A3)

Having in mind classical mechanics, one sees that the original physical laws lie in the time dependence of the transition probability and, because of (A2), in the time dependence of the U of the unitary pair. The simplest differential equation for U_t is a linear differential equation, making sure that all U_t are unitary:

(A3) The time dependence of unitary pairs is given by

$$i\hbar \frac{\partial U_t}{\partial t} = H_t \circ U_t$$

H_t are (essential) self-adjoint operators defined on the natural Hilbert space in the decision.

Planck's constant $\hbar$ is not essential in this axiom; one could have incorporated it in H_t. As this equation is written just in quantum mechanics in this form, whereas in classical mechanics it is often not mentioned, I left Planck's constant on the left side together with the time derivative. The flows of classical mechanics generated by the acceleration field $b_t(r, v)$ have as operators

$$H_t = -\frac{i}{2}\hbar \frac{\partial b_t^k(r, v)}{\partial v^k} - i\hbar v^k \frac{\partial}{\partial r^k} - i\hbar b_t^k(r, v) \frac{\partial}{\partial v^k}$$

One sees that Planck's constant and the imaginary unit drop out of the time evolution equation given in (A3). The factor $\frac{1}{2}$ in the first term depends on the function $f(x) = x^2$ in (A2). Taking for classical mechanics the natural Banach spaces $\mathscr{L}^p$ with $1 \leq p \neq 2$, we find that the formulas for the transition probabilities would not change, but for a norm-preserving operator of the time evolution, in the first term one has to change the factor $\frac{1}{2}$ into $1/p$ (Gerlich, 1991, p. 48; Gerlich, 1987*b*, pp. 102, 234, 239).

In this case, for the corresponding probability density

$$\rho_t = (\psi_t \bar{\psi}_t)^{p/2}, \qquad \psi_t = N_t \psi_0$$

one gets the generalized Liouville equation

$$\frac{\partial \rho_t}{\partial t} + \frac{\partial}{\partial r^k}(v^k \rho_t) + \frac{\partial}{\partial v^k}(b_t^k \rho_t) = 0$$

which is a continuity equation independent of p. This example shows that the use of the $\mathscr{L}^p$-spaces with $p \neq 2$ alone does not force the time evolution equation to be nonlinear. In Mielnik (1974) the $\mathscr{L}^p$-spaces with $p \neq 2$ are a consequence of the special form of the nonlinear Schrödinger equations proposed by him.

In this context one could give an answer to the often discussed question of why a real Hilbert space is not sufficient in quantum theory. For classical mechanics the imaginary unit i and Planck's constant $\hbar$ drop out of the equation describing the time evolution. The time evolution is of such a special form that one could use the generalized Liouville equation instead of the equation given in (A3). But in quantum mechanics, the imaginary unit does not drop out of the equation describing the time evolution. The continuity equation for the probability density is no longer equivalent to the equation of (A3). With real vector spaces, the formulation of the time evolution operators H of quantum mechanical systems would be much more complicated, introducing an R^2-valued $\mathscr{L}^2$-space or a direct sum of two R^1-valued $\mathscr{L}^2$-spaces, which could be written as $\mathbb{C}^1$-valued $\mathscr{L}^2$-spaces.

6. AXIOMS OF MECHANICS

The last axioms now following treat the quantum mechanical and classical mechanical description of mass points separately. Like the first Newtonian axiom, the first typically quantum mechanical axiom describes a free mass point:

(AQS1) The space of events of the space measurements of a free particle is R^3 with the Lebesgue measure μ_L on the Lebesgue σ-algebra $\mathbf{A}_L$. The self-adjoint operator of the time evolution for this system is given by

$$H = -\frac{\hbar^2}{2m}\Delta, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

defined on a suitable dense linear subspace of the natural Hilbert space $\mathscr{L}^2(\mu, R^3)$.

The corresponding axiom of classical mechanics of mass points reads as follows.

(AMN1) The space of events of the space and velocity measurements of a free particle is R^6 with the Lebesgue measure μ_L on the Lebesgue σ-algebra $\mathbf{A}_L$. The self-adjoint operator of the time evolution is given by

$$H = -i\hbar \mathbf{v} \cdot \nabla, \qquad \mathbf{v} \cdot \nabla = v_x \frac{\partial}{\partial y} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$$

defined on a suitable dense linear subspace of the natural Hilbert space $\mathscr{L}^2(\mu_L, R^6)$.

With the gauge technique first introduced by Weyl (1931, 1977) one can find an Ansatz for a particle in an external electromagnetic field. A gauge transformation in the Hilbert space in the decision

$$U_E \psi_t|_{\mathbf{r}} = e^{i\alpha(\mathbf{r},t)} \psi_t(\mathbf{r}), \qquad \alpha(\mathbf{r}, t) \in R^1$$

does not change the transition probabilities. This results in a class of equivalent operators $H_E = U_E \circ H \circ U_E^\dagger$ with external fields describing the same time evolution of the transition probability (Gerlich, 1987*b*, p. 188). One gets an inequivalent operator, substituting the gauge fields by $A(1) = (\mathbf{A}_1 - V)$

$$\tilde{H}_E = \frac{1}{2m} (-i\hbar\nabla - \mathbf{A}) \cdot (-i\hbar\nabla - \mathbf{A}) + V$$

if in four-dimensional space-time the field $A(1)$ has a nonvanishing alternating differential $F(2) = \nabla \wedge A(1) \neq 0$.

One can identify this alternating differential $F(2)$ with the electromagnetic field; then the latter equation $F(2) = \nabla \wedge A(1)$ is the homogeneous part of Maxwell's equations. This suggests the following axiom:

(AQS2) The operator of the time evolution of a particle of charge Ze and mass m in an external electromagnetic field with the potential $(\mathbf{A}, -V)$ is given by

$$H = \frac{1}{2m} (-i\hbar\nabla - Ze\mathbf{A}) \cdot (-i\hbar\nabla - ZeA) + ZeV$$

defined on a suitable dense linear subspace of the natural Hilbert space defined in (AQS1) (with the same space of events).

One can use the gauge technique together with the axiom (AMN1) of classical mechanics. *Surprisingly, one does not get the analogous axiom*

(AMN2) of classical mechanics, showing that this axiom should not be interpreted as an approximation of (AQS2):

(AMN2) The operator of the time evolution in an external acceleration field is given by

$$H=-i\frac{\hbar}{2}\frac{\partial b_t^k(r,v)}{\partial v^k}-i\hbar v^k\frac{\partial}{\partial r^k}-i\hbar b_t^k(r,v)\frac{\partial}{\partial v^k}$$

defined on a suitable dense linear subspace of the natural Hilbert space defined in (AMN1) (with the same space of events).

The first Newtonian axiom defines space-time in the mathematical model on which the force fields of the second Newtonian axiom are defined. Similarly, the axioms (AQS1) and (AMN1) define the spaces on which the operators of (AQS2) and (AMN2) with external fields are defined. The axioms (AMN1) and (AMN2) remain unchanged for a nonradiating relativistic electron substituting the acceleration field

$$\mathbf{b}=-\frac{e}{m_0}\left(1-\frac{v^2}{c^2}\right)^{1/2}\left(\mathbf{E}-\frac{1}{c^2}\mathbf{v}\mathbf{v}\cdot\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$$

E and **B** are the external electromagnetic fields. For a radiating electron, the space of events has to be enlarged, as such a system is commonly described by differential equations of third order for the space coordinates. The relativistic form of the corresponding quantum mechanical axioms (Dirac's equation) changes significantly with respect to (AQS1) and (AQS2), formally being more like the classical mechanical form:

(AQD1) The space of events of the space measurements of a free electron is R^3 with the Lebesgue measure μ_L on the Lebesgue-σ-algebra $\mathbf{A}_L$. The operator of the time evolution for this system is given by

$$H=m_0c^2\beta-ic\hbar\boldsymbol{\alpha}\cdot\nabla$$

defined on a suitable dense linear subspace of the $\mathbb{C}^4$-valued natural Hilbert space $\mathscr{L}^2(\mu_L, R^3, \mathbb{C}^4)$ being isomorphic to the $\mathbb{C}^2$-valued natural Hilbert space

$$\mathscr{L}^2\left(\mu_{AP}\otimes\mu_L,\left\{-\frac{\hbar}{2},\frac{\hbar}{2}\right\}\times R^3\right)\oplus\mathscr{L}^2\left(\mu_{AP}\otimes\mu_L,\left\{-\frac{\hbar}{2},\frac{\hbar}{2}\right\}\times R^3\right)$$

(μ_{AP} is the counting measure on $\{-\hbar/2, \hbar/2\}$) and the corresponding complex-valued natural Hilbert space [for mathematical details see Gerlich (1987*b*)].

Applying the gauge technique of Weyl, one gets an Ansatz for an electron in an external electromagnetic field:

(AQD2) The operator of the time evolution of an electron in an external electromagnetic field $(\mathbf{A}, -V)$ is given by

$$H = m_0 c^2 \beta - ic\hbar \boldsymbol{\alpha} \cdot \nabla + ec\boldsymbol{\alpha} \cdot \mathbf{A} - eV$$

defined on a suitable dense linear subspace of the natural Hilbert space defined in (AQD1) (with the same space of events).

These axioms treat particles in an external field. Of course one cannot make much physics with such systems. In the next step one should describe interacting particles. Considering classical mechanics, one would like to formulate (AMN2) at once for systems of interacting particles formally only changing the dimension of the space of events. In the case of Schrödinger's equation (AQS2) this would also be possible for simple interactions with some technical effort (fermions, bosons). In both cases one has some difficulties with the relativistic form. Therefore, I would like to show how this step can be done naturally in the present approach, when one takes into account the historical development of physics.

Methodologically, in the development of physics, the first important step was Newton's discovery that the (inertial) masses were the sources of the gravitation field leading to the Newtonian gravitational interaction of mass points. The next step was describing the constraints analytically (d'Alembert's, Gauss's principle). Not considering the spaces of events, formally, the form of the second Newtonian axiom (AMN2) remains unchanged. This changes when electric currents are interpreted as moving electric particles with magnetic interaction. These forces do not act in the direction of the two particles and depend on the velocities [Clausius potential (Clausius, 1879)]. With his hypothesis of electrons, H. A. Lorentz added a mechanical velocity of particles to Maxwell's theory and could explain many phenomena of optics, but new difficulties arose (Poincaré, 1900): The equations were no longer relativistic using Galilei's transformations. Second, the third Newtonian law was no longer valid. These things required the repair of the famous ether; Lorentz had to invent many hypotheses to save his electron theory until he succeeded in finding a relativistic form with the Lorentz transformation (Lorentz, 1904) named by Poincaré (1905, 1906). In the course of time the third Newtonian axiom was no longer taken too seriously, but the radiation of point particles as constituents of atoms remained a problem.

Usually one describes the radiating electron with differential equations of third order. The new formulation of (AMN2) for such equations of motion can be given without special difficulties. But the problem of the stability of nonradiating atoms was solved in another way. Schrödinger (1926) noticed when he published his time-dependent Schrödinger equation [in our formulation this is contained in the axioms (AQS1) and (AQS2)], that for the density (*Gewichtsfunktion*)

$$\rho_t = \psi_t \bar{\psi}_t, \qquad \psi_t = U_t \psi_0$$

one has a continuity equation in common three-dimensional space integrating all but one of the particle variables. Taking the current density corresponding to this continuity equation as a source of the electromagnetic field, it was understandable why the atoms were not radiating in the ground state: the charge density became independent of time. Therefore, Schrödinger (1926) talks about "a certain return to electrostatic and magnetostatic models of atoms." Radiation of an atom is nothing else but an electromagnetic field at a distance produced by the atom. Therefore, it is clear that the current density calculated from ψ_t should be the source of the electromagnetic fields.

This is in agreement with our formulation of the axioms. One can derive from each axiom (AQS2), (AMN2), and (AQD2) continuity equations [generalized Liouville equations (Gerlich, 1987*b*, pp. 148, 195, 318)]. As is known, the inhomogeneous part of Maxwell's equations is equivalent to a continuity equation, disregarding differentiability conditions. From a logical standpoint, therefore, in this way, the ψ-functions should be the sources of the electromagnetic interactions (as it was analogously with the gravitation). Along this line Barut and Kraus (1976, 1977) worked out an electromagnetic theory of elementary particles. If one looks upon the ψ-function as a mathematical device to produce the electromagnetic fields, it does at a first glance seem to be in contradiction to our axiom (A2), where the ψ-functions are a mathematical device to calculate transition probabilities. In the sense of axiom (A2), $\psi_t \bar{\psi}_t$ is the probability density measuring the space coordinates of an electron. But these interpretations can easily be connected to answer the question of when is it possible to measure a probability distribution of the space coordinates of particles. This is impossible for particles interacting with nothing (recall the search for neutrinos!). Only if $\psi_t(\mathbf{r})$ has something to do with the interaction of the particle, can one hope to find with repeated measurements a frequency distribution for interaction processes (a density curve). In this case the black interaction point has to be much smaller than the range where $\psi_t \bar{\psi}_t$ is essentially unequal to zero if the probability density should be testable by experiments. The probability interpretation necessarily demands that in some way the interaction can be calculated with ψ-functions.

Thus, stressing the analogous historical development of classical and quantum mechanics, one should not overlook the important difference in the fact that in classical mechanics the sources of the interactions are described with the space and velocity coordinates, not with the ψ-functions. This is a consequence of the difference between the axioms (AQ) and (AM).

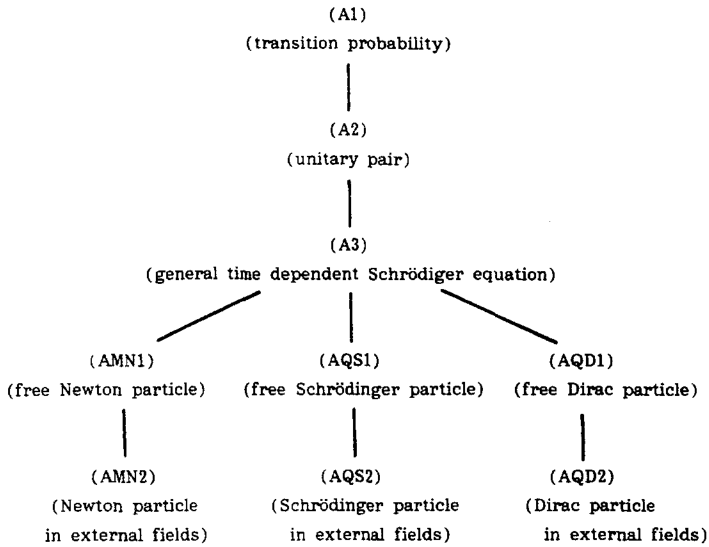
7. GENERALIZED QUANTUM THEORIES

Systems of axioms should give hints for possible generalizations and appreciable modifications of a theory. If I try to show such modifications, this is of course rather subjective and speculative. In some sense the given generalizations are necessary. Therefore, they can serve to indicate the critical points of our given system of axioms.

The essential difference from other systems of axioms of quantum theory known to me is the structure of the events in the condition in (A1). Together with (A2), this is a generalization of the projection postulate valid also for the continuous spectrum of the events in the condition, respectively a more concrete interpretation of the conventional formula of the transition probability of quantum theory $|\langle\phi|\psi\rangle|^2$. The change of the representation, the S-matrix theory, and conventional classical statistical mechanics fit this generalized formulation very well. This is no longer true if one could only describe discrete events in the condition with this formula. Of course, this does not mean that such a structure *must* be listed in a system of axioms.

But if one takes this structure of the events in the condition seriously, one should incorporate it with all mathematical consequences. One such consequence would be taking an equivalent finite measure on the events in the condition (with the same sets of measure zero) instead of only σ-finite measures. Thus, all measurable sets (events) could be taken as elements of the natural vector spaces and could be used in the formula of (A2) for $U(\chi_A)$. If the measure ν is not finite, for finite $\nu(A)$, the characteristic function for the complement of A is not an element of the natural vector space. Introducing finite measures for the events in the condition changes the formula of the unitary pair (Gerlich, 1987*b*, p. 229). Making convergent the partition function of the ideal hydrogen system, such a manipulation with the *a priori* measure is already in use (Gerlich, 1977, p. 92). In this sense such a generalization does not seem to be serious. But with our formula of unitary pairs, not only are counting measures admitted for the events in the condition, but also the Lebesgue measure for space events. Substituting the Lebesgue measure by an equivalent finite measure, one gets problems describing the homogeneity of the space.

A schematic representation of our system of axioms (Scheme I) (Gerlich, 1987*a*) suggests combining the columns of the last two lines in a single formulation. This was successfully done with the Newtonian and Schrödinger axioms (Gerlich, 1987*b*, p. 188; Gerlich, 1987*a*; Dietert, 1990), taking the spaces of events from (AMN1) also for (AQS1/2) and adding the



Scheme I

operators H with small modifications. One gets a generalization of quantum mechanics containing classical mechanics, or vice versa a generalization of classical mechanics containing quantum mechanics. The physical consequences are studied in Dietert (1990).

REFERENCES

Barut, A. O., and Kraus, J. (1976). *Journal of Mathematical Physics*, **17**, 506–508.

Barut, A. O., and Kraus, J. (1977). *Physical Review D*, **16**, 161–163.

Barut, A. O., and Meystre, P. (1984). *Physics Letters*, **105A**, 458–462.

Bauer, H. (1974). *Wahrscheinlichkeitstheorie und Grundzüge der Masstheorie*, de Gruyter, Berlin.

Belinfante, F. J. (1973). *A Survey of Hidden-Variables Theories*, Pergamon Press, Oxford.

Bell, J. S. (1965). *Physics*, **1**, 195.

Clausius, R. (1879). *Die mechanische Wärmetheorie II, Die mechanische Behandlung der Electricität*, 2nd ed., Vieweg Verlag, Braunschweig.

Davies, E. B. (1976). *Quantum Theory of Open Systems*, Academic Press, London.

Dietert, T. (1990). *Phasenraum Schrödingergleichungen, Quantenmechanik und klassische Mechanik in einer umfassenden Theorie*, COGNOS-Institut Dreyer, Braunschweig, Germany.

Gerlich, G. (1974). *Reelle Massmannigfaltigkeiten zur Beschreibung physikalischer Zusammenhänge*, Habilitationsschrift der TU Braunschweig, Germany.

Gerlich, G. (1977). *Eine neue Einführung in die statistischen und mathematischen Grundlagen der Quantentheorie*, Vieweg-Verlag, Braunschweig, Germany.

Gerlich, G. (1980). *Klassiche Feldtheorie* (*Elektrodynamik*), SS 1980, Manuskript zur Vorlesung, Sonderveröffentlichungen, TU Braunschweig, Germany.

Gerlich, G. (1981). *Erkenntnis*, **16**, 335–338.

Gerlich, G. (1983). *Quantentheorie I, II*, WS 1982/83, SS 1983, Manuskripte zur Vorlesung, Sonderveröffentlichungen, TU Braunschweig, Germany.

Gerlich, G. (1987*a*). The role of probability and statistics in physics, in *Probability and Bayesian Statistics*, R. Viertl, ed., Plenum Press, New York.

Gerlich, G. (1987*b*). *Quantentheorie*, Manuskript zur Vorlesung, Sonderveröffentlichungen, TU Braunschweig, Germany.

Gerlich, G. (1991). *Klassische Mechanik*, WS 1979/80, *Mechanik*, WS 1985/86, (3rd ed.), Manuskripte zur Vorlesung, Sonderveröffentlichungen, TU Braunschweig, Germany.

Gudder, S. P. (1970). *Journal of Mathematical Physics*, **11**, 431–436.

Hellwig, G. (1964). *Differentialoperatoren der mathematischen Physik*, Springer-Verlag, Berlin.

Hilbert, D., Nordheim, L., and von Neumann, J. (1927). Über die Grundlagen der Quantenmechanik, *Mathematische Annalen*, **98**, 1–30 [Lecture notes of D. Hilbert (WS 1926/27) according to L. Nordheim and J. von Neumann]; reprinted in J. von Neumann, *Gesammelten Werke*, Vol. 1, pp. 104–133.

Holevo, A. S. (1973). *J. Multivariate Analysis*, **3**, 337–394.

Hopf, E. (1937/1970). *Ergodentheorie*, Springer-Verlag, Berlin.

Jauch, J. M. (1973). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Kingman, J. F., and Taylor, S. J. (1966). *Introduction to Measure and Probability*, Cambridge University Press, Cambridge.

Kolmogoroff, A. N. (1933/1977). *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer, Berlin.

Lorentz, H. A. (1904). *Proceedings Academy of Sciences of Amsterdam*, **6**, 809.

Ludwig, G. (1976). *Einführung in die Grundlagen der Theoretischen Physik*, Vol. 3, *Quantentheorie*, Vieweg, Braunschweig, Germany.

Mackey, G. W. (1963). *The Mathematical Foundations of Quantum Mechanics*, Benjamin, New York.

Mielnik, B. (1974). *Communications in Mathematical Physics*, **37**, 221–256.

Nelson, E. (1967). *Dynamical Theories of Brownian Motion*, *Mathematical Notes*, Princeton University Press, Princeton, New Jersey.

Pancović, V. (1989). *Physics Letters A*, **137**, 158–160.

Piron, C. (1976). *Foundation of Quantum Physics*, Benjamin, Reading, Massachusetts.

Poincaré, H. (1900). Über die Beziehungen zwischen der experimentellen und der mathematischen Physik, Übersetzung de Vortrags, gehalten auf dem internationalen Physikerkongress zu Paris am 7. August 1900, *Physikalische Zeitschrift*, **2**(11), 166–171; (12) 182–186; (13) 196–201.

Poincaré, H. (1905). *Comptes Rendus de l'Academie des Sciences*, **40**, 1504–1508.

Poincaré, H. (1906). *Rendiconti Circ. Matematica Palermo*, **21**, 129–175.

Schrödinger, E. (1926). *Annalen der Physik*, **81**, 109.

Von Neumann, J. (1932/1968). *Mathematische Grundlagen der Quantenmechanik*, Springer-Verlag, Berlin.

Weyl, H. (1931/1977). *Gruppentheorie und Quantenmechanik*, Wissenschaftliche Buchgesellschaft, Darmstadt, Germany.